K25P 0958

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (C.B.C.S.S. – O.B.E. – Regular) Examination, April 2025
(2023 Admission)
MATHEMATICS/MATHEMATICS (Multivariate Calculus and Mathematical Analysis, Modelling and Simulation, Financial Risk Management)
MSMAT04E06/MSMAF04E07 : Fourier and Wavelet Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any five** questions. **Each** question carries **4** marks. **(5×4=20)**

1. Define conjugate reflection. Suppose z, w $\in l^2(\mathbb{Z}_N)$, prove that $z * \tilde{w}(k) = \langle z, R_k w\rangle$ for any $k \in \mathbb{Z}$.

2. Suppose N is even, say N = 2M, $z \in l^2(\mathbb{Z}_N)$ and $w \in l^2(\mathbb{Z}_{N/2})$. Prove that $D(z) * w = D(z * U(w))$.

3. Prove that trigonometric system is an orthonormal set in $L^2([-\pi, \pi))$.

4. Suppose H is a Hilbert space and $\{a_j\}_{j\in\mathbb{Z}}$ is an orthonormal set in H. Prove that $\{a_j\}_{j\in\mathbb{Z}}$ is a complete orthonormal set if and only if $f = \sum_{j\in\mathbb{Z}} \langle f, a_j\rangle a_j$ for all $f \in H$.

5. Suppose z, $w \in l^2(\mathbb{Z})$, prove that $U(z * w) = U(z) * U(w)$.

6. Suppose f, $g \in L^1(\mathbb{R})$ and $\hat{f}, \hat{g} \in L^1(\mathbb{R})$. Prove that $\langle \hat{f}, \hat{g}\rangle = 2\pi\langle f, g\rangle$.

PART – B

Answer **any three** questions. **Each** question carries **7** marks. **(3×7=21)**

7. Suppose $M \in \mathbb{N}$, $N = 2M$ and $w \in l^2(\mathbb{Z}_N)$. Prove that $\{R_{2k}w\}_{k=0}^{M-1}$ is an orthonormal set with M elements if and only if $\left|\widehat{w}(n)\right|^2 + \left|\widehat{w}(n+M)\right|^2 = 2$ for n = 0, 1, ..., M – 1.

8. Let $u \in l^2(\mathbb{Z}_4)$ be such that $\hat{u} = (1, \sqrt{2}, i, 0)$. Find some $\hat{v}$ such that $\{v, R_2v, u, R_2u\}$ is an orthonormal basis for $l^2(\mathbb{Z}_4)$.

P.T.O.

9. Suppose $T : L^2([-\pi, \pi)) \to L^2([-\pi, \pi))$ is a bounded, translation invariant linear transformation. Then prove that for each $m \in \mathbb{Z}$, there exists $\lambda_m \in \mathbb{C}$ such that $T(e^{im\theta}) = \lambda_m e^{im\theta}$.

10. Suppose $z \in l^2(\mathbb{Z})$ and $w \in l^1(\mathbb{Z})$. Then prove that $z * w \in l^2(\mathbb{Z})$ and $\| z * w \| \leq \| w \|_1 \| z \|$.

11. Suppose $f, g \in L^1(\mathbb{R})$. Prove that

   i) $\int_{\mathbb{R}} |f(x-y)g(y)|\, dy < \infty$ for a.e $x \in \mathbb{R}$.

   ii) $f * g \in L^1(\mathbb{R})$ with $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

PART – C

Answer **any three** questions. **Each** question carries **13** marks. **(3×13=39)**

12. Suppose $M \in \mathbb{N}$ and $N = 2M$. Let $u, v \in l^2(\mathbb{Z}_N)$. Prove that $B = \{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$ if and only if the system matrix $A(n)$ of u and v is unitary for each $n = 0, 1, ..., M-1$.

13. a) Define trigonometric system. Prove that the trigonometric system is complete in $L^2([-\pi, \pi))$.

   b) Suppose $\{a_j\}_{j \in \mathbb{Z}}$ is an orthonormal set in a Hilbert space H. Let $S_A = \left\{ \sum_{j \in \mathbb{Z}} z(j) a_j : z = (z(j))_{j \in \mathbb{Z}} \in l^2(\mathbb{Z}) \right\}$. Define $P_S(f) = \sum_{j \in \mathbb{Z}} \langle f, a_j \rangle a_j$. Prove that $P_S : H \to S$ is linear.

14. a) Suppose $f : [-\pi, \pi] \to \mathbb{C}$ is continuous and bounded. If $\langle f, e^{in\theta} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta = 0$ for all $n \in \mathbb{Z}$, prove that $f(\theta) = 0$ for all $\theta \in [-\pi, \pi]$.

   b) Define Fourier transform on $l^2(\mathbb{Z})$. Prove that Fourier transform is one to one.

15. Suppose $w \in l^1 \mathbb{Z}$ and $z \in l^2(\mathbb{Z})$. Then prove that

   i) $(z * w)^\wedge(\theta) = \hat{z}(\theta)\widehat{w}(\theta)$ a.e

   ii) $z * w = w * z$

16. Suppose $f, g \in L^2(\mathbb{R})$. Prove the following statements :

   i) $\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle$

   ii) $\|\hat{f}\| = \sqrt{2\pi}\,\|f\|$

   iii) $\langle \hat{f}, \hat{g} \rangle = \dfrac{1}{2\pi}\langle f, g \rangle$

   iv) $\|\hat{f}\| = \dfrac{1}{\sqrt{2\pi}}\|f\|$.